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The toroidal bubble

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It has been observed by Walters & Davidson (1963) that release of a mass of gas in water sometimes produces a rising toroidal bubble. This paper is concerned with the history of such a bubble, given that at the initial instant the motion is irrotational everywhere in the water. The variation of its overall radius a with time may be predicted from the vertical impulse equation, and it should be possible to make the same prediction by equating the rate of loss of combined kinetic and potential energy to the rate of viscous dissipation. This is indeed seen to be the case, but not before it is recognized that in a viscous fluid vorticity will continually diffuse out from the bubble surface, destroying the irrotationality of the motion, and necessitating an examination of the distribution of vorticity. The impulse equation takes the same form as in an inviscid fluid, but the energy equation is severely modified. Other results include an evaluation of the effect of a hydrostatic variation in bubble volume, and a prediction of the time which will have elapsed before the bubble becomes unstable under the action of surface tension.

1. Introduction

The toroidal bubble is a phenomenon first observed by Walters & Davidson (1963) in the course of their investigation into the development of an initially spherical bubble of air in water. Its form is that of a vortex-ring with a buoyant air core, and the circulation associated with the vortex-ring is produced not by viscous forces, but by the inviscid process of bubble formation, so that the flow around it is initially irrotational everywhere. The present paper seeks to trace the history of a toroidal bubble after its formation. We shall not only calculate how its overall properties (radius and velocity of rise, for example) vary with time, but we shall also examine the flow around it, and inquire into the stability of the bubble, in order to predict if and when it will eventually break up.

Our notation is introduced by means of figure 1. The cross-section of the air-core is assumed for the moment to be a circle of radius b, and the curve joining the centres A of all such circles is itself a circle with radius a and centre instantaneously at the fixed point O. The plane of this circle is horizontal, and the axis Oz is the upwards vertical. Two sets of co-ordinates will be used in the course of this paper: cylindrical polar co-ordinates (r, ϕ, z), where r and z are defined in figure 1, and ϕ is the azimuthal angle; and the set (s, χ, ϕ), where (s, χ) are polar co-ordinates in a meridional plane, centred at the point A (values of $s \cos \chi$ less than $(-a)$ are prohibited). Let the circulation about the bubble be $2\pi\Gamma$, defined as the circulation round a large material circuit threading the ring along Oz (we assume that the fluid through which the bubble rises is effectively unbounded), and assume

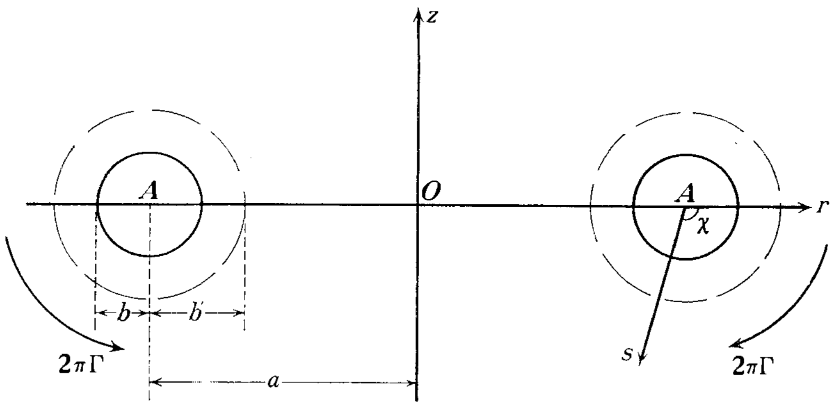


FIGURE 1. Meridional section of the toroidal bubble, summarizing the notation used in the text.

that Γ remains constant with time. If the ambient fluid is inviscid, this follows from Kelvin's circulation theorem, and the assumption is still true in the viscous case, at least until any vorticity which may be generated at the bubble surface has diffused to the axis; it will be shown that only a negligible amount of vorticity ever reaches the axis. From their measurements of a, b and the velocity of rise U of the bubble, Walters & Davidson observed that Γ is indeed approximately constant, provided that Lamb's (1932, p. 241) formula for the velocity of translation of a vortex-ring in terms of its circulation and core dimensions is valid here. The value thus found for Γ was close to that which Walters & Davidson predicted in terms of g (the gravitational acceleration) and V (the bubble volume), viz.

$$\Gamma = \frac{3}{2\pi}(gV)^{\frac{1}{2}}. \tag{1.1}$$

We shall further assume that b is at all times much smaller than a.

The overall properties of the bubble can be most easily deduced from the vertical impulse equation (as for any buoyant vortex-ring, see Turner 1957), which may be written

$$dP/dt = F, \tag{1.2}$$

where F is the resultant force on the water in the vertical direction, due in this case to the buoyancy of the bubble, less the rate of momentum loss suffered in a viscous wake, if any. P is the vertical 'fluid impulse' associated with the vortex-ring at a given time t, and is given by the formula (2.1) below (Lamb 1932, p. 214 *et seq.*); P may be regarded as the net impulse which would be required to set up the flow instantaneously from rest.

It is instructive also to deduce the bubble properties from the energy equation, for it is only there that it becomes clear how great the effect on the flow of a non-zero viscosity actually is. The energy equation is

$$\frac{d}{dt}(T+\Omega) = -D, \tag{1.3}$$

where T and Ω are the kinetic and potential energies of the system, and D is the total rate of loss of energy by viscous dissipation, zero in an ideal fluid, but not zero in a real fluid, even when the flow is irrotational. The rate of change of potential energy, neglecting the small contribution from surface tension (that it is small for typical bubbles may be verified *a posteriori*), is given by

$$\frac{d\Omega}{dt} = \frac{d}{dt}(g\rho Vh) = -g\rho VU, \tag{1.4}$$

where ρ is the density of the ambient fluid, h is the depth of the bubble beneath a fixed reference level, U is its velocity of rise, and V (its volume) is assumed constant. The remaining quantities in (1.2) and (1.3) and U, cannot be determined until we have an acceptable model for the flow round the bubble.

Section 2 deals with the case when the ambient fluid is inviscid, and it is shown that formulae derived for a homogeneous vortex-ring (from Lamb 1932, §§162–3), and giving P, T, U in terms of Γ, a and b may be used with only slight modification (the assumption $b/a \ll 1$ is here essential). Expressions for a and U as functions of time t are deduced: a ultimately increases as $t^{\frac{1}{2}}$, and U decreases as $t^{-\frac{1}{2}}\log t$. It is also shown that if the flow is assumed to remain approximately irrotational in a viscous fluid, so that Lamb's formulae may again be used almost as they stand, the equations of impulse and energy yield conflicting results. Presumably, therefore, the flow does not remain approximately irrotational, and vorticity is continually created at the bubble surface, whence it diffuses out into the fluid. Section 3 contains an analysis of that diffusion, and it is shown that the vorticity distribution becomes approximately Gaussian, with an effective radius b' which also increases as $t^{\frac{1}{2}}$. Section 4 demonstrates that formulae similar to the inviscid ones may still be used for P, T, U, etc., but with b' replacing b, and the solution for a as a function of t turns out to be the same as in the inviscid case, while U decreases as $t^{-\frac{1}{2}}$.

In §5 we compute the effect of a hydrostatic variation of the volume V as the bubble rises, and it turns out not to be entirely negligible from an experimental point of view. (All predictions in this paper are formulated with reference to a typical bubble from Walters & Davidson's experiments, to underline the fact that they apply to a realizable situation.) Section 6 analyses the stability of the bubble, assuming that the curvature of the air core is irrelevant in a first approximation, when $b/a \ll 1$. Initially, the circulation at the bubble surface is rapid enough to counteract the destabilizing influence of surface tension, but it is reduced continually by the action of viscosity, so that at a predictable moment surface tension will become dominant, and instability will occur. The complete life-cycle of the toroidal bubble, from formation to disintegration, will thus have been described in detail.

2. Development of the bubble in an inviscid ambient fluid.

The irrotational flow with circulation $2\pi\Gamma$ round a toroidal bubble is instantaneously the same as that round the core of a homogeneous vortex-ring of the same dimensions and circulation, as long as the distribution of azimuthal vorticity in that core is suitably chosen. The choice must be made so that the bubble surface becomes a stream surface of the combined flow, separating rotational from irrotational motion. The shape of the bubble surface itself, however, is determined by its being a surface of constant pressure (incorporating hydrodynamic, hydrostatic, and surface tension components), so the problem is an implicit one, whose solution will yield both the shape of the bubble surface and the equivalent distribution of vorticity in the core. Assuming that the flow is axisymmetric, the impulse P_1 and kinetic energy T_1 of the combined system may then be calculated by the methods of Lamb (1932, §§162–3), whence the impulse P and kinetic energy T of the irrotational part of the flow are obtained by subtracting the momentum P_0 and kinetic energy T_0 of the fluid undergoing rotational motion in the core.

Lamb's formulae for the impulse and kinetic energy of an arbitrary axisymmetric distribution of azimuthal vorticity ω are

$$P_1 = \pi\rho \iint \omega r^2 \, dr \, dz, \quad T_1 = -\pi\rho \iint \omega\psi \, dr \, dz, \tag{2.1}$$

where ψ is the Stokes stream function of the flow, ρ is the fluid density, and the integration is taken over the meridional half-plane $0 \leqslant r < \infty$, $-\infty < z < \infty$. As an example, Lamb calculates T_1 for a vortex-ring with a circular core (radius b) of uniform vorticity ($2\Gamma/b^2$), and obtains

$$T_1 = 2\pi^2\rho a\Gamma^2\left[\log\frac{8a}{b} - \frac{7}{4} + O\left(\frac{b^2}{a^2}\log\frac{8a}{b}\right)\right] \tag{2.2}$$

(the error term is estimated by extending Lamb's analysis to a higher order in b/a). For the same distribution of vorticity we obtain

$$P_1 = 2\pi^2\rho a^2\Gamma\left[1 + O\left(\frac{b^2}{a^2}\right)\right]. \tag{2.3}$$

Now the cross-section of the core of a toroidal bubble will not be exactly circular, but if b/a is small, where b is now the mean radius of the core, the boundary of the cross-section will be of the form

$$s = s_0(\chi) \equiv b\left[1 + f_1(\chi)F_1\left(\frac{b}{a}\right) + O\left(\frac{b^2}{a^2}\log\frac{8a}{b}\right)\right], \tag{2.4}$$

where (s, χ) are polar co-ordinates in the meridional plane, $f_1(\chi)$ is a periodic function, of order one, with period 2π, and $F_1(b/a)$ is independent of χ and is assumed to be at most of order $(b/a \log 8a/b)$. This assumption is based on the actual shape of a 'hollow vortex-ring' in the absence of gravity and surface tension, as calculated by Hicks (1884)

$$s_0(\chi) = b\left[1 - \frac{3b}{8a}\left(\log\frac{8a}{b} - \frac{5}{8}\right)\cos\chi + O\left(\frac{b^2}{a^2}\log\frac{8a}{b}\right)\right]; \tag{2.5}$$

the effects of gravity and surface tension are assessed in §4 (they tend to make the core cross-section more nearly circular). Again, the equivalent vorticity distribution cannot be exactly uniform, the simplest dynamically possible distribution is one where ω/r is uniform, and in general may be written

$$\omega(s, \chi) = \frac{2\Gamma}{b^2}\left[1 + f_2(\chi)F_2\left(\frac{b}{a}\right) G_2\left(\frac{s}{b}\right) + O\left(\frac{b^2}{a^2}\log\frac{8a}{b}\right)\right], \tag{2.6}$$

where f_2, F_2 have properties similar to f_1, F_1 and G_2 is some function of s/b, of order one. But even with these more general expressions for ω and the cross-section boundary $s_0(\chi)$, Lamb's methods yield the same formulae (2.2) and (2.3) for T_1 and P_1. The only quantity which does depend, to the first order in $b/a \log 8a/b$, on the forms of $s_0(\chi)$ and ω is the velocity of translation of the vortex-ring, U, which may be calculated from the condition that there is no normal velocity across what was the bubble surface. If, for instance, the core cross-section is taken to be circular, and the quantity ω/r uniform inside it, then U is given by Lamb's formula

$$U = \frac{\Gamma}{2a}\left[\log\frac{8a}{b} - n + O\left(\frac{b^2}{a^2}\log\frac{8a}{b}\right)\right], \tag{2.7}$$

with $n = \frac{1}{4}$, but if $s_0(\chi)$ and ω take the more general forms (2.4) and (2.6), the value of n is in general altered (Hicks (1884) gives the value $n = \frac{1}{2}$ for his 'hollow vortex-ring').

The vertical momentum P_0 of the fluid in the core of the equivalent vortex-ring is approximately the momentum of a quiescent ring of fluid, of volume V, travelling with velocity U. Now $V = 2\pi^2 ab^2$, so that

$$P_0 \approx \rho V U \approx 2\pi^2\rho a^2\Gamma\frac{1}{2}\frac{b^2}{a^2}\log\frac{8a}{b},$$

whence the impulse of the flow round the toroidal bubble is

$$P = P_1 - P_0 = 2\pi^2\rho a^2\Gamma\left[1 + O\left(\frac{b^2}{a^2}\log\frac{8a}{b}\right)\right], \tag{2.8}$$

the second term of which is assumed to be negligible. Similarly, the kinetic energy T_0 of the fluid in the core is approximately that of a cylinder of fluid of the same mass (ρV) and the same radius (b), rotating with angular velocity Γ/b^2 and translating with linear velocity U, and is approximately given by

$$T_0 \approx \tfrac{1}{2}\pi^2\rho a\Gamma^2.$$

Thus
$$T = T_1 - T_0 = 2\pi^2\rho a\Gamma^2\left[\log\frac{8a}{b} - 2 + O\left(\frac{b^2}{a^2}\log\frac{8a}{b}\right)\right]. \tag{2.9}$$

We may now calculate the time variation of the overall radius of the bubble, from both the impulse and the energy equations. In the impulse equation (1.2), the only contribution to the force F is the buoyancy force $g\rho V$, so that from (1.2) and (2.8) we obtain

$$\frac{d}{dt}(2\pi^2\rho a^2\Gamma) = g\rho V,$$

whence
$$a^2 = a_0^2 + \frac{gV}{2\pi^2\Gamma}t = a_0^2 + C_1 t \quad \text{(say)}, \tag{2.10}$$

where a_0 is the value of a at time $t = 0$. Equation (2.10) is accurate as long as

$$\frac{b^2}{a^2}\log\frac{8a}{b} \ll 1. \tag{2.11}$$

To the same accuracy, the energy equation (1.3), together with (1.4), (2.7) and (2.9), gives

$$\frac{\dot{a}}{a}\left(\log\frac{8a}{b}-1\right)-\frac{\dot{b}}{b}=\frac{gV}{4\pi^2\Gamma}\frac{1}{a^2}\left(\log\frac{8a}{b}-n\right);$$

but $V = 2\pi^2ab^2$ is assumed to be constant, so $\dot{b}/b = -\dot{a}/2a$, and hence

$$a\dot{a}\left(\log\frac{8a}{b}-\frac{1}{2}\right)=\frac{gV}{4\pi^2\Gamma}\left(\log\frac{8a}{b}-n\right),$$

which again leads to (2.10), as long as $n = \frac{1}{2}$ (although even without this, the leading terms in the equation give (2.10) still).

Knowing a, we may now calculate the variation of U with time, from (2.7). In particular, at large times, we have

$$U \approx \frac{2\Gamma}{a}\log\frac{8a}{b} \approx \frac{3\pi\Gamma^{\frac{3}{2}}}{(2gVt)^{\frac{1}{2}}}\log\left(\frac{16gV^{\frac{1}{3}}t}{2^{\frac{1}{3}}\pi^{\frac{2}{3}}\Gamma}\right); \tag{2.12}$$

but Γ is approximately given in terms of g and V by (1.1), so that U is given in terms of g, V and t alone by

$$U \approx \frac{3^{\frac{5}{2}}}{4(\pi t)^{\frac{1}{2}}}(gV)^{\frac{1}{4}}\log\left(\frac{16(4\pi)^{\frac{1}{3}}gt}{3V^{\frac{1}{6}}}\right). \tag{2.12$'$}$$

It is interesting to compare this with the constant velocity of rise of spherical cap bubbles, where $U \propto g^{\frac{1}{2}}V^{\frac{1}{6}}$ (Taylor & Davies 1950).

3. The effect of viscosity on the flow

If we were now to suppose that the flow round a toroidal bubble in a *viscous* fluid at large Reynolds number were essentially irrotational, except perhaps in certain thin shear layers, we would be able to use the same formulae for P, T and U as for an inviscid fluid. The impulse equation would be unaltered, leading once more to (2.10). The energy equation, however, would have to contain a term describing the viscous dissipation of energy D, cf. (1.3). In order to calculate D, let us approximate the flow round the toroidal bubble by the irrotational flow round a cylindrical bubble of radius b; Lamb's (1932, p. 241, equation (4)) formula shows that this is a good approximation in a region close to the bubble surface, and, as will be evident, most of the dissipation occurs in that region. The dissipation in a two-dimensional swirling motion, with tangential velocity $v(s)$ outside the cylinder $s = b$ (in plane polar co-ordinates s, χ) is

$$D' = \mu\int_0^{2\pi}\int_b^{\infty}\left(\frac{dv}{ds}-\frac{v}{s}\right)^2 s\,ds\,d\chi \quad \text{per unit length,} \tag{3.1}$$

where μ is the dynamic viscosity of the fluid. For irrotational flow of circulation $2\pi\Gamma$, $v = \Gamma/s$, yielding

$$D' = \frac{4\pi\mu\Gamma^2}{b^2}\text{ per unit length.}$$

Multiply this by $2\pi a$, the overall circumference of the bubble, and obtain for the total dissipation in the irrotational flow round a toroidal bubble:

$$D = \frac{8\pi^2\mu a\Gamma^2}{b^2}\left[1+O\left(\frac{b}{a}\right)\right], \tag{3.2}$$

where the error is consequent upon the neglect of the curvature of the bubble core. In fact, an exact calculation of the dissipation in the flow round a vortex-ring is possible (see Pedley 1966, chapter IV, § 6, where the calculation is performed for the special case of a circular core of uniform vorticity) and it results precisely in equation (3.2), justifying the above approximation.

However small μ may be, the continual increase of a/b^2 ultimately makes dissipation significant and the energy equation (1.3) yields a possible steady solution, for which

$$D = g\rho VU,$$

and T, a, U, b are all constant in time. This conflicts with (2.10) and indicates that the flow does not remain essentially irrotational. Clearly, all vorticity creation must take place at the bubble surface, where the condition of zero tangential stress is violated by irrotational flow. What we now have to investigate is the manner in which that vorticity diffuses out from the bubble surface, and its effect on the impulse and energy equations.

The diffusion of vorticity from the bubble surface

In order to make the problem tractable, let us assume that the cross-section of the bubble *is* circular, so that in the co-ordinate system (s, χ, ϕ), the bubble surface has equation $s = b$ (i.e. we neglect $b/a \log 8a/b$ compared with unity). We seek the ϕ-component of vorticity, ω, as a function of space and time, under the conditions that it is initially zero everywhere, and that the tangential stress on $s = b$ is zero. The problem is complicated by the fact that the overall radius a of the bubble must be permitted to vary with time; if a increases with time, azimuthal vortex lines are continually stretched, so that their strength increases, and the rate of diffusion is correspondingly enhanced; however, b simultaneously decreases (to maintain the constant bubble volume), and this shrinking of the radial length scale is accompanied by an inwards convection of vorticity, inhibiting the diffusion. The combination of these two effects must be taken into account.

The vorticity equation in terms of the non-inertial co-ordinate system (s, χ, ϕ) is extremely complicated, so, as a first approximation, let us neglect the curvature of the bubble (and hence of the vortex lines), as in the above calculation of dissipation. This approximation requires not only that b/a be very small, but also that the vorticity be effectively confined to a region surrounding the bubble in which s/a is everywhere small. The problem is now two-dimensional, and we use cylindrical polar co-ordinates (s, χ, ζ), with corresponding velocity components (u, v, w), in which the bubble surface is the infinite circular cylinder $s = b(t)$, and the initial flow (for $s > b$) is that of a potential vortex of circulation $2\pi\Gamma$. The whole system is uniformly stretched in the ζ-direction, in such a way as to keep the bubble volume $(2\pi^2ab^2)$ constant. Thus, since a measure of the axial length scale is the overall circumference $2\pi a$ of the toroidal bubble, we have

$$\frac{\partial w}{\partial \zeta} = \frac{\partial \dot{\zeta}}{\partial \zeta} = \frac{\dot{a}}{a} = -\frac{2\dot{b}}{b}. \tag{3.3}$$

By symmetry the ζ-component of vorticity, ω, is a function only of s, and the corresponding component of the vorticity equation is

$$\frac{\partial \omega}{\partial t}+u\frac{\partial \omega}{\partial s} = \omega\frac{\partial w}{\partial \zeta}+\nu\left(\frac{\partial^2\omega}{\partial s^2}+\frac{1}{s}\frac{\partial \omega}{\partial s}\right), \tag{3.4}$$

where ν is the kinematic viscosity of the fluid and $\partial w/\partial\zeta$ is given by (3.3). The radial velocity u is determined by the continuity equation

$$\frac{1}{s}\frac{\partial}{\partial s}(su)+\frac{\partial w}{\partial \zeta} = 0,$$

whose solution, from (3.3) and the condition $u = \dot{b}(t)$ on $s = b(t)$, is

$$u = \dot{b}s/b. \tag{3.5}$$

Equation (3.4) for ω now becomes

$$\frac{\partial}{\partial t}(s\omega)+\frac{\dot{b}}{b}\frac{\partial}{\partial s}(s^2\omega) = \nu\frac{\partial}{\partial s}\left(s\frac{\partial \omega}{\partial s}\right); \tag{3.6}$$

but

$$\omega = \frac{1}{s}\frac{\partial}{\partial s}(sv), \tag{3.7}$$

so (3.6) has a first integral

$$\frac{\partial}{\partial t}(sv)+\frac{\dot{b}}{b}s\frac{\partial}{\partial s}(sv) = \nu s\frac{\partial}{\partial s}\left[\frac{1}{s}\frac{\partial}{\partial s}(sv)\right]+f(t), \tag{3.8}$$

where $f(t)$ is a function of integration. The boundary conditions on v are

$$\left.\begin{aligned} sv = \Gamma, \quad & t = 0, \quad s > b(0), \\ sv \to \Gamma, \quad & s/b \to \infty, \quad t \geqslant 0, \\ \frac{\partial}{\partial s}\left(\frac{v}{s}\right) = 0, \quad & s = b(t), \quad t > 0, \end{aligned}\right\} \tag{3.9}$$

(this last being the condition of zero tangential stress on the bubble surface), and the second of these shows that $f(t)$ is identically zero.

The problem can be further simplified if we refer it to fixed boundaries by means of the transformations

$$x = \frac{s}{b(t)}, \quad \tau = \int_0^t \frac{\nu\,dt}{b^2(t)}, \quad C = \frac{sv}{\Gamma}. \tag{3.10}$$

The equations and boundary conditions finally become

$$\left.\begin{aligned} \frac{\partial C}{\partial \tau} &= \frac{\partial^2 C}{\partial x^2}-\frac{1}{x}\frac{\partial C}{\partial x}, \\ C = 1, \quad & \tau = 0, \quad x > 1, \\ C \to 1, \quad & x \to \infty, \quad \tau \geqslant 0, \\ \frac{\partial C}{\partial x} = 2C, \quad & x = 1, \quad \tau > 0, \end{aligned}\right\} \tag{3.11}$$

which is a typical diffusion problem with fixed boundaries. The exact solution may be obtained in the form of an integral, by the use of Laplace transform techniques (see Pedley 1966, chapter v), but for our purposes it is sufficient to have the well-known asymptotic solution for large τ, viz:

$$C \sim 1-\exp(-x^2/4\tau) \quad \text{as} \quad \tau \to \infty, \tag{3.12}$$

which is in fact accurate everywhere to within 2 % if $\tau \geqslant 1$. The asymptotic form of the vorticity distribution may be obtained from (3.7), (3.10) and (3.12) and is given by

$$\omega \sim \frac{\Gamma}{2b^2\tau} \exp\left(-\frac{s^2}{4b^2\tau}\right), \tag{3.13}$$

provided that $s/a \ll 1$ everywhere in the region of appreciable vorticity. The vorticity distribution is thus Gaussian, with an effective radius $b'(t)$ given by

$$b'^2 = 4b^2\tau = \frac{4\nu}{a}\int_0^t a\,dt, \tag{3.14}$$

if the condition of constant bubble volume is applied. The conditions for validity of this solution are $b'/a \ll 1$ and $\tau \geqslant 1$. Before we can verify that they are satisfied, we must know a as a function of t, so in the next section we assume the Gaussian distribution (3.13) in order to calculate a, and then check that the procedure is self-consistent. We notice here that if, at large times, a^2 increases as $C_1 t$, cf. (2.10), then b'^2 increases as $\frac{8}{3}\nu t$ (from 3.14), and the condition $b'/a \ll 1$ reduces to $(8\nu/3C_1)^{\frac{1}{2}} \gg 1$, which depends on the given physical parameters of the bubble.

4. Development of the bubble in a viscous ambient fluid

The distribution of vorticity generated at the bubble surface has been shown to tend rapidly to the form given by (3.13), under the condition $b'/a \ll 1$, i.e.

$$\omega = \frac{2\Gamma}{b'^2}\exp\left(-\frac{s^2}{b'^2}\right)\left[1+O\left(\frac{b'}{a}\log\frac{8a}{b'}\right)\right]. \tag{4.1}$$

Lamb's formulae (2.1) for the impulse and kinetic energy of a distribution of ring vorticity may be applied to the distribution (4.1) as much as to those of § 2, although this time the s-integrations must extend from zero to infinity (the contribution from $s/b' \gtrsim 3$ is negligible, and that from $s < b$ is a small correction, tending to zero as $t \to \infty$ and $b/b' \to 0$, which will also be ignored; i.e. we assume $P = P_1$ and $T = T_1$). If the error term in (4.1) is periodic in χ (as it must be on physical grounds), the results of the calculations are

$$P = 2\pi^2\rho a\Gamma\left[1+O\left(\frac{b'}{a}\right)\right] \tag{4.2}$$

and

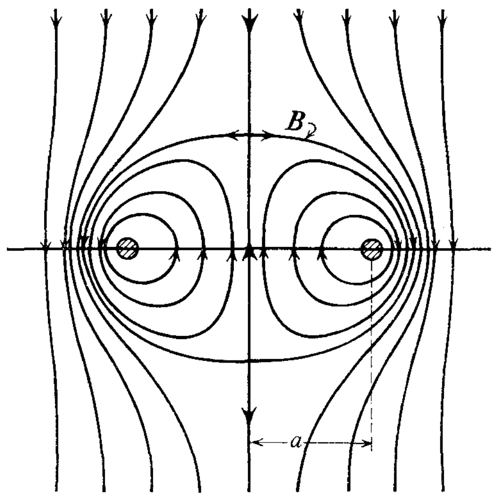
$$T = 2\pi^2\rho a\Gamma^2\left[\log\frac{8a}{b'} - N + O\left(\frac{b'}{a}\log\frac{8a}{b'}\right)\right], \tag{4.3}$$

where

$$N = 2+\tfrac{1}{2}(\log 2-\gamma_0) \approx 2{\cdot}06$$

(γ_0 is Euler's constant), a value little different from the 2 obtained in § 2 on the assumption of an almost circular distribution of almost constant vorticity. The velocity of translation of the vorticity distribution (4.1) depends, as for the distributions of § 2, on the exact condition employed to define it. All methods give the formula (2.7), with b' in place of b, but with various values of n, and possibly an increased error term.

Because equation (4.2) for the impulse has the same leading terms as (2.8), the impulse equation (1.2) for the bubble in a viscous fluid will be exactly the same



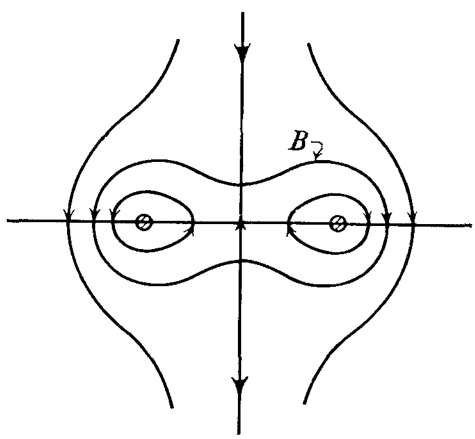


FIGURE 2. Meridional section of the streamline pattern in the irrotational flow past a vortex-ring of small core cross-section (shaded area): $b'/a \ll 1$. Axes fixed in ring. (a) $b'/a > 1/35$; (b) $1/35 < b'/a < 1/86$.

as in the inviscid case, provided that there is no momentum loss in a viscous wake. To see that the actual momentum loss is indeed negligible, consider the streamlines of the irrotational flow round a vortex ring whose vorticity distribution is characterized by a radius b', such that b'/a is small. Qualitatively, they are as shown in figure 2a, at least when b'/a is greater than about 1/35 (if b'/a is less than about 1/86, the region of closed streamlines becomes a ring, and there exist intermediate values of b'/a for which this region is simply connected, but re-entrant, as in figure 2*b*). The shaded portions of figure 2 are circles of radius b', within which most of the vorticity lies when its distribution is approximately Gaussian. Now, the Gaussian approximation is asymptotically valid at large times ($\tau \gtrsim 1$), but in any case the vorticity falls off exponentially from the bubble surface at all times, so that a negligibly small amount of vorticity ever reaches the edge of the region of closed streamlines (i.e. the boundary B of the fluid carried along by the vortex-ring, see figure 2) which is characterized by the dimension a. Now all vorticity diffusing across B is convected around by the ambient flow, and is swept off downstream in a narrow wake (see Moore 1963, for a detailed analysis of this effect when the region of closed streamlines is a spherical bubble). Since in our case very little vorticity ever reaches B, it follows that very little is swept off in the wake, whence the flow outside B is very nearly irrotational, and the momentum deficit across the wake is negligible. This argument depends on the assumption $b'/a \ll 1$, which we must verify later. Note that the fact that there is no drag term in the impulse equation does not necessarily imply that the viscous dissipation in the flow is negligible, although very often (as in the flow past a spherical bubble) the two are connected. In our case, the dissipation takes place primarily inside the region of closed streamlines, and the viscous forces do not contribute to a momentum deficit across the wake. In addition, the exponential decay of the vorticity away from the bubble surface applies as well to its reaching the axis of the ring as to its reaching B, and so the assumption of constant Γ, discussed in § 1, is justified.

We thus see that the impulse equation, and hence formula (2.10) for a as a function of time, is the same in the viscous case as in the inviscid case. We can now calculate b' as a function of t from (3.14), and it is given exactly by

$$b'^2 = \frac{8\nu a_0^2}{3C_1}\left[\left(1+\frac{C_1 t}{a_0^2}\right)^{\frac{3}{2}}-1\right]\left(1+\frac{C_1 t}{a_0^2}\right)^{-\frac{1}{2}},$$

which for large values of $C_1 t/a_0^2$ reduces to

$$b'^2 = \tfrac{8}{3}\nu t, \tag{4.4}$$

as predicted at the end of § 3. We may also calculate the formula equivalent to (2.12) for U as a function of t, from (2.10), (4.4) and the leading term of (2.7); and at large times it becomes

$$U = \frac{2\pi\Gamma^{\frac{3}{2}}}{(2gVt)^{\frac{1}{2}}}\log\frac{12gV}{\pi^2\nu\Gamma} = \frac{3^{\frac{3}{4}}(gV)^{\frac{1}{4}}}{2(\pi t)^{\frac{1}{2}}}\log\frac{8(gV)^{\frac{1}{2}}}{\pi\nu}, \tag{4.5}$$

when Γ is given by (1.1). Note that $U \propto t^{-\frac{1}{2}}$, which agrees with the similarity solution of Turner (1957) for buoyant vortex-rings, and should be contrasted with the inviscid results (2.12) and (2.12′).

We can also write down the energy equation for the bubble in a viscous fluid, and show that it no longer conflicts with the impulse equation. The dissipation D may be calculated from (2.13), with $v(s)$ given according to the Gaussian vorticity distribution as

$$v(s) = \frac{\Gamma}{s}\left[1-\exp\left(-\frac{s^2}{b'^2}\right)\right].$$

The integrations are again simple, and the leading term in the result is

$$D = 4\pi^2\mu a\Gamma^2/b'^2.$$

Thus the energy equation (1.3) this time gives

$$\frac{d}{dt}\left[2\pi^2\rho a\Gamma^2\left(\log\frac{8a}{b'}-N\right)\right] = \frac{g\rho V\Gamma}{2a}\left(\log\frac{8a}{b'}-n\right)-\frac{4\pi^2\mu a\Gamma^2}{b'^2},$$

which may be rearranged as follows

$$\left(\frac{da^2}{dt}-C_1\right)\left(\log\frac{8a}{b'}-N+1\right) = \frac{2a^2}{b'^2}\left[\frac{d}{dt}(\tfrac{1}{2}b'^2)-2\nu\right]+(N-n-1)C_1$$
$$= \frac{-a^2}{2t}+(N-n-1)C_1,$$

when b'^2 is given by (4.4). The asymptotic solution of this is exactly $a^2 = C_1 t$ if $n = N-\frac{3}{2} \approx 0{\cdot}56$, but for any n the asymptotic solution has the form $a^2 = C_1' t$, and the maximum error, with $n = 1$ say, for the particular bubble described below, is less than 1 %. Thus the energy equation yields almost exactly the same answer as the impulse equation, and we have clearly taken all the important physical effects into account.

The value of a given by (2.10), and the corresponding value of U obtained from the exact forms of a and b', are plotted as functions of t in figure 3, for a typical set of bubble constants. Figure 4 is a graph of a against the height to which the bubble has risen since its release. It demonstrates that the bubble rises conically (radius $\propto$ height) as we would expect from Turner's similarity arguments again. (The broken curves in figure 3 are those obtained when hydrostatic variations in the bubble volume V are taken into account, see §5.) The values used for Γ, V, etc., are taken from a typical bubble in Walters & Davidson's experiments, and are:

$$\Gamma = 50\,\text{cm}^2\,\text{sec}^{-1},\quad V = 21\,\text{cm}^3,\quad a_0 = 5{\cdot}0\,\text{cm},\quad b_0 = 0{\cdot}46\,\text{cm}.$$

We also require the physical constants

$$g = 980\,\text{cm}\,\text{sec}^{-2},\quad \nu = 0{\cdot}011\,\text{cm}^2\,\text{sec}^{-1}.$$

The use of the above formulae for a and U depends on the fact that $b'/a \ll 1$. In the typical example being considered, b'/a initially has the value $b_0/a_0 \approx 0{\cdot}092$, and tends rapidly to the value $(8\nu/3C_1)^{\frac{1}{2}} \approx 0{\cdot}037$, so it is always reasonably small. The use of the Gaussian distribution of vorticity requires $\tau \gtrsim 1$, which here means $t \gtrsim 10$ sec, and the particularly simple formula (4.4) for b' is valid if $C_1 t/a_0^2 \gg 1$, or $t \gg 1{\cdot}2$ sec. Thus after about 10 sec in the life of this typical bubble, all the simplest formulae are valid; up to that time the more exact formulae must be employed.

Finally, we may show that the effect of gravity and surface tension on the shape of the bubble cross-section does not exceed that of the dynamic pressure, so that the perturbation from circular form is at most of order $(b/a)\log(8a/b)$, as in the estimate (2.4) for $s_0(\chi)$. The maximum variation of dynamic pressure round a toroidal bubble of circular cross-section (using Bernoulli's equation and a refinement of Lamb's formula for the stream function near the bubble surface) is of order $\rho\Gamma^2/ab$ ($\approx 1000\,\text{dynes}\,\text{cm}^{-2}$ here), while the maximum variation in hydrostatic pressure is of order $\rho g b$ ($\approx 500\,\text{dynes}\,\text{cm}^{-2}$), and in the apparent

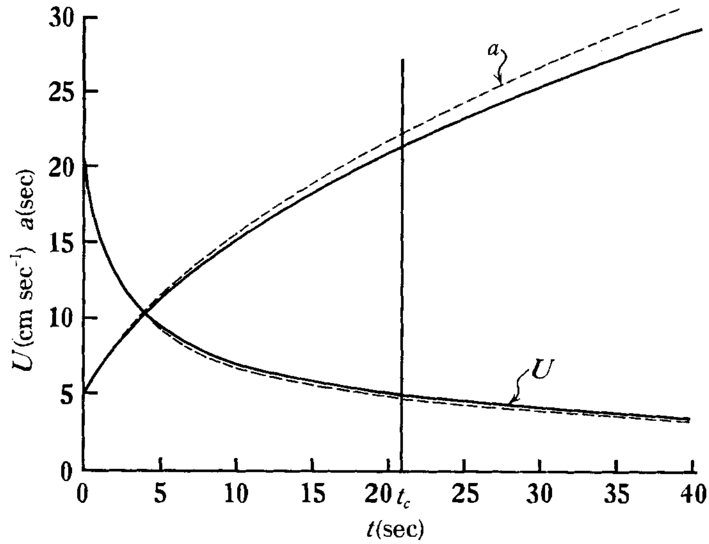


FIGURE 3. Graphs of bubble radius a and velocity U against time t. ———, constant volume; – – –, volume increasing.

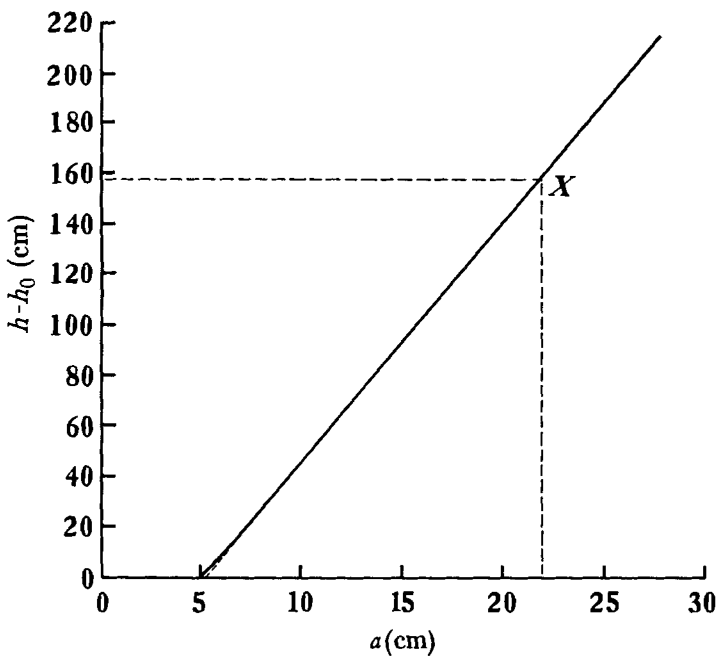


FIGURE 4. Graph of bubble radius against height, demonstrating the conical shape of the region swept out by the bubble. X marks the point at which instability occurs.

pressure due to surface tension γ is of order γ/b ($\approx 150\,\text{dynes}\,\text{cm}^{-2}$). Thus our assumption in §2 was justified, especially since gravity tends to elongate the bubble cross-section horizontally, while the dynamic pressure tends to elongate it vertically (cf. Hicks's formula (2.5)), and anyway surface tension tends always to restore circularity.

5. The effect of a hydrostatic variation of bubble volume

So far the bubble volume has been assumed to be constant. In a real experiment, however, the volume of an air bubble increases as the bubble rises, owing to the decrease in hydrostatic pressure, and this will have some effect on the predicted variation of the bubble radius, say, with time. Assume that the mixture of air and water-vapour in the bubble is a perfect gas at constant temperature, let h be the depth of the bubble below the free surface of the tank when its volume is V, and let $p_0 = g\rho d$ be the atmospheric pressure, where d is taken to be 1030 cm. Then the bubble pressure is $p = g\rho(d+h)$, so that Boyle's Law pV = constant becomes

$$(d+h)V = (d+h_0)V_0,$$

where h_0 and V_0 are the values of h and V at time $t = 0$. The impulse equation $dP/dt = g\rho V$ therefore reduces to

$$\frac{da^2}{dt} = C_1 \frac{1+h_0/d}{1+h/d} \quad \text{where} \quad C_1 = \frac{gV_0}{2\pi^2\Gamma}, \tag{5.1}$$

which is not as readily integrable as before because h varies according to

$$dh/dt = -U,$$

and U itself depends on a and t. However, a numerical integration of (5.1) is simple to perform, leading to a solution which, for our typical bubble, is compared in figure 3 with the constant volume solution. (For the case shown, h_0 was taken to be $\frac{1}{4}d$, but varying this quantity has no distinguishable effect on the solution.) The curves for a and U deviate from the constant volume curves by just less than 7 % in a time of 40 sec. This would be noticeable in an experiment, although by that time the bubble (a) would have become unstable (see § 6)—the time when instability would be expected to occur is marked t_c on figure 3—and (b) would have reached the top of the tank. The curve for a against (h_0-h) is not, on the scale drawn, distinguishable from its constant volume counterpart.

6. Stability

The toroidal bubbles observed by Walters & Davidson were evidently stable. However, any toroidal bubble must after a certain time become unstable since the destabilizing influence of surface tension continually increases as b decreases, and the stabilizing effect of the circulation continually decreases as the velocity at the bubble surface decreases from the action of viscosity. The physical quantities which might be important in the stability analysis are surface tension, viscosity, and the basic flow. It will become apparent that the only relevant parameter of the basic flow is the velocity at the bubble surface. As in the calculation of the dissipation, therefore, the curvature of the core may be neglected, and the basic flow may be taken to be two-dimensional, consisting of a tangential velocity $v(s)$ (where s is the radial co-ordinate) outside the infinitely long cylindrical 'bubble' whose boundary $s = b$ is a free surface under the action of surface tension γ. From §3, v is known to be given approximately by

$$v(s) \approx \frac{\Gamma}{s}\left[1-\exp\left(\frac{-3s^2}{8\nu t}\right)\right], \tag{6.1}$$

at least for times of the order of or larger than $3b^2/8\nu$. We assume that the critical time we are looking for *is* at least of this order, and must verify it afterwards.

Pedley (1967) has investigated the stability to small disturbances of precisely this basic state. The analysis ignores viscosity, and is therefore likely to be valid only if disturbance time scales are much shorter than the viscous diffusion time $3b^2/8\nu$. With this proviso, however, the results may be applied here directly. It is established that a necessary and sufficient condition for the system to be stable to an axisymmetric disturbance of wave-number k, as long as $d(s^2v^2)/ds$ is positive (which it is), is that

$$1-k^2b^2-\frac{\rho b v^2(b)}{\gamma} < 0. \tag{6.2}$$

It is also demonstrated that for a non-axisymmetric disturbance whose azimuthal wave-number is the integer n, a sufficient condition for stability is

$$1-n^2-k^2b^2-\frac{\rho b v^2(b)}{\gamma} < 0, \tag{6.3}$$

provided that v/s is essentially constant near $s = b$. Thus if

$$\frac{\rho b v^2(b)}{\gamma} > 1, \tag{6.4}$$

the flow must be stable, and even when that condition is not satisfied, it is only axisymmetric disturbances ($n = 0$) which can be unstable. At the time when $bv^2(b)$ has decreased so far that (6.4) ceases to be valid, axisymmetric instability will set in.

Thus when v is given by (6.1), the critical time t after which the bubble will become unstable is given by

$$\frac{\rho b v^2(b)}{\gamma} \equiv \frac{\rho\Gamma^2}{b\gamma}\left[1-\exp\left(\frac{-3b^2}{8\nu t_c}\right)\right] = 1. \tag{6.5}$$

If t_c is large compared with $3b^2/8\nu$, so that the basic flow *is* approximately solid-body rotation near $s = b$, then (6.5) may be expanded in powers of $(3b^2/8\nu t_c)$, and the first approximation gives

$$t_c \approx \frac{3\Gamma}{8\nu}\left(\frac{\rho b^3}{\gamma}\right)^{\frac{1}{2}}. \tag{6.6}$$

For the typical bubble of §4 (bearing in mind the variation of b with time) t_c is approximately 21 sec. The condition $t_c \gg 3b^2/8\nu$ requires

$$W = \frac{\rho\Gamma^2}{b\gamma} \gg 1,$$

i.e. the Weber number W (comparing inertial and surface tension forces in the basic flow) must be large. In our case, taking $\gamma = 74\,\text{dynes}\,\text{cm}^{-1}$, $W \approx 70 \gg 1$. The only other assumption to be checked is that, for the purposes of the stability analysis, the basic flow is approximately steady; that is, a typical disturbance time scale t_1 is small compared with $3b^2/8\nu$. The obvious choice for t_1 comes from the equation of motion for axisymmetric disturbances (Pedley 1967, equation (5.1), $t_1 \sim 1/\sigma$) where in general the order of magnitude of t_1 must be given by

$$\frac{1}{t_1^2} = \sigma^2 \sim \frac{1}{s^3}\frac{d}{ds}[s^2 v^2(s)] = O\left(\frac{\Gamma^2}{b^4}\right).$$

Hence the condition $b^2/\nu \gg t_1$ becomes $\Gamma/\nu \gg 1$, and requires a large Reynolds number in the basic flow. In our case $\Gamma/\nu \approx 5 \times 10^3$, and is indeed large.

The calculated value of t_c is of course only approximate, but there would be no point using, for example, a more accurate expression for $v(b)$, without at the same time making corrections for the lack of core circularity, and for the time-dependence of the basic stability problem, and these modifications would require excessive effort for a small return. The quoted formula for t_c will certainly give an answer of the right order of magnitude. Note that at the time this bubble does become unstable, the deviations in the values of a and U due to the variation of bubble volume are still only 4 % (see figure 3) which would be almost negligible in an experiment. The value of a at time t is (from figure 3) just over 21 cm, and from figure 4 we see that the bubble will then have risen 160 cm, which is within experimental possibility, but is greater than the depth of Walters & Davidson's tank (3 ft.). Even the very smooth bubble shown in plate 4 of their paper would have become unstable (according to this theory) if the experiment had been performed in a six- or seven-foot tank.

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